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Non-classical Study on the Simultaneous Rational Approximation

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ABSTRACT

This study is placed in the framework of Internal Set Theory (Nelson, 1977). Real numbers $(\xi_i)_{i=1,2,\dots,k}$ are called simultaneously approximable in the infinitesimal sense, if for every positive infinitesimal ε, there exist rational numbers $(\frac{p_i}{q})_{i=1,2,\dots,k}$ such that

$$\begin{cases} \xi_i = \dfrac{p_i}{q} + \varepsilon £_i \\ \varepsilon q \simeq 0 \end{cases} ;\ i = 1,2,\dots,k,$$

where $(£_i)_{i=1,2,\dots,k}$ are limited numbers. Let $(\xi_0, \xi_1, \dots, \xi_\omega)$ be a system of reals, with ω unlimited. In this paper, we will give a necessary condition for which $(\xi_i)_{i=0,1,\dots,\omega}$ are simultaneously approximable in the infinitesimal sense. The converse of this condition is also discussed.

Keywords: INTERNAL set theory, simultaneous rational approximation, infinitesimal sense.

1. INTRODUCTION

1.1 Elementary Nonstandard Notions

We recall some definitions and facts from nonstandard analysis, real numbers, and sets that will be used in the proof of the main result. For more details, see Diener, 1960; 1995, Lutz and Goze , 1981 and Nelson, 1977.

(a) A real number x is called *unlimited* if its absolute value $|x|$ is larger than any standard integer n. So a nonstandard integer ω is also an unlimited real number.

(b) A real number ε is called *infinitesimal* if its absolute value $|\varepsilon|$ is smaller than $\frac{1}{n}$ for any standard n.

(c) A real number r is called *limited* if is not unlimited and *appreciable* if it is neither unlimited nor infinitesimal.

(d) Two real numbers x and y are *equivalent* (written $x \simeq y$) if their difference $x - y$ is infinitesimal.

(e) We distinguish two types of formulas: Formulas which do not contain the symbol "st" (for standard) are called *internal*, and formulas which do contain the symbol "st" are called *external*.

(f) We call internal any set defined using an internal formula.

(g) We call external any subset of an internal set defined using of an external formula for which a classical theorem at least is in default.

Examples 1.1. From Diener et Reeb, 1960 in page 52, we have

1. Let ε be a positive real number (infinitesimal or not). The following sets are internal.

   $[1-\varepsilon, 1+\varepsilon]$, $\{x \in \mathbb{R} \,;\, \varepsilon x \geq 1\}$, and $\left\{\frac{n}{\varepsilon} \,;\, n \in \mathbb{N}\right\}$.

2. Let ε be an infinitesimal positive real number. The set

$$\{x \in \mathbb{R} \,;\, x + \varepsilon \simeq x\}$$

is equal to $\mathbb{R}$. *i.e.*, it is internal. However, the set

$$\{x \in \mathbb{R} \,;\, x \simeq 0\} \tag{1}$$

is external. In fact, if the set of (1) is internal, then it has the least upper bound a; which is neither infinitesimal nor appreciable (If a is infinitesimal, $2a$ and $3a$ are also. If a is appreciable, $\frac{a}{2}$ is also).

That is, the Least Upper Bound Principle "*A nonempty set of reals which is bounded above has the least upper bound*" is in default.

3. Let $\mathbb{N}^\sigma$ be the set of limited (standard) positive integers, then $\mathbb{N}^\sigma$ is external. In fact, if it is not we can apply the Principle of Mathematical Induction:

   - Since 1 is limited, then $1 \in \mathbb{N}^\sigma$.
   - If $s \in \mathbb{N}^\sigma$, then $s + 1 \in \mathbb{N}^\sigma$.

   Therefore $\mathbb{N}^\sigma = \mathbb{N}$, which is impossible because there are unlimited positive integers.

Lemma 1.2 (Robinson's Lemma). *If $(u_n)_{n\geq 0}$ is a sequence such that $u_n \simeq 0$ for all standard n, there exists an unlimited N such that $u_n \simeq 0$ for all $n \leq N$.*

Also, in this paper, we need to the following notions (see Cutland, 1983, Diener, 1995 and Van den Berg, 1992).

Definition 1.3. Let X be a standard set, and let $(A_x)_{x\in X}$ be an internal family of sets.

1. A union of the form $G = \bigcup_{stx\in X} A_x$ is called a *pregalaxy*; if it is external G is called a *galaxy*.
2. An intersection of the form $H = \bigcap_{stx\in X} A_x$ is called a *prehalo*; if it is external H is called a *halo*.

Example 1.4. We have

(a) $\mathbb{N}^\sigma$ is a galaxy.
(b) $hal(0) = \{x \in \mathbb{R}\,;\ x \simeq 0\}$ is a halo.

Theorem 1.5. *No halo is a galaxy.*

Definition 1.6 (Shadow of a set). *The shadow of a set A, denoted by $°A$, is the unique standard set whose standard elements are precisely those whose halo intersects A.*

Theorem 1.7 (Cauchy's Principle). *No external set is internal.*

For example, let ω be an unlimited positive integer. The shadow of $\left(\frac{1}{\omega}, \frac{2}{\omega}, \dots, \frac{\omega-1}{\omega}, \frac{\omega}{\omega}\right)$ is equal to $[0,1]$. Moreover, we see that $e^n < \omega$ for every

standard positive integer n. From Cauchy's Principle there exists an unlimited integer n_0 which satisfies the previous inequality.
In this work limited numbers are denoted by £ and infinitesimal numbers are denoted by ε or ϕ.

1.2 Some Classical Results on the Simultaneous Rational Approximation

We present some well known results on the simultaneous approximation of k numbers $\xi_1, \xi_2, \dots, \xi_k$ by fractions $\frac{p_1}{q}, \frac{p_2}{q}, \dots, \frac{p_\omega}{q}$. These results were announced by Dirichlet's Theorem (Schmidt, 1980 in page 27)), Kronecker's Theorem (Hardy and Wright, 1960 in page 382), and many others.

Theorem 1.8 (Dirichlet's Theorem). *Let k be a positive integer, and let $\xi_1, \xi_2, \dots, \xi_k$ be reals. For any integer $Q > 1$, we can find positive integers $q, p_1, p_2, \dots, p_k$ such that*

$$1 \le q < Q^k \text{ and } |q\xi_i - p_i| \le \frac{1}{Q}; \text{ for } i = 1,2, \dots, k.$$

Theorem 1.9 (Hardy and Wright, 1960, page 170). *If $\xi_1, \xi_2, \dots, \xi_k$ are any real numbers, then the system of inequalities*

$$\left|\xi_i - \frac{p_i}{q}\right| < \frac{1}{q^{1+\mu}}, \mu = \frac{1}{k} \text{ ; for } i = 1,2, \dots, k.$$

has at least one solution. If one ξ_i at least is irrational, then it has an infinity of solutions.

Theorem 1.10 (Hardy and Wright, 1960, page 170). *Given $\xi_1, \xi_2, \dots, \xi_k$ and any positive ε, we can find an integer q so that $q\xi_i$ differs from an integer, for every i, by less than ε.*

Definition 1.11 (Schmidt, 1962)**.** *A set of numbers $\xi_1, \xi_2, \dots, \xi_r$ is linearly independent if no linear relation:*

$$a_1\xi_1 + a_2\xi_2 + \dots + a_r\xi_r = 0,$$

with integer coefficients, not all zero, holds between them.

Theorem 1.12 (Kronecker's Theorem). *If $\xi_1, \xi_2, \dots, \xi_k$ are linearly independent, $\alpha_1, \alpha_2, \dots, \alpha_k$ are arbitrary, and Q and ε are positive, then there are integers*

$$q > Q, p_1, p_2, \dots, p_k$$

such that $|q\xi_i - p_i - \alpha_i| < \varepsilon$, *for* $i = 1,2,\dots,k$.

1.3 How to give the infinitesimal sense to the simultaneous rational approximation?

Let k be a positive integer and let $(\xi_1, \xi_2, \dots, \xi_k)$ be a system of reals. From Dirichlet's Theorem, for any integer $Q > 1$, the reals $(\xi_i)_{i=1,2,\dots,k}$ are simultaneously approximable by the rational numbers $(\frac{p_i}{q})_{i=1,2,\dots,k}$, with an error less than $\frac{1}{qQ}$. That is,

$$\xi_i = \frac{p_i}{q} + e_i, \text{ with } |e_i| \le \frac{1}{qQ} \text{ and } 1 \le q < Q^k \; ; \; i = 1,2,\dots, \; k.$$

For every positive infinitesimal ε, we can choose Q such that $\varepsilon Q^k \simeq 0$, which implies $\varepsilon q \simeq 0$. Thus,

$$\begin{cases} \xi_i = \frac{p_i}{q} + \varepsilon\gamma_i \\ \quad \varepsilon q \simeq 0 \end{cases} \quad \text{with} \quad |\gamma_i| \le \frac{1}{\varepsilon q Q} \; ; \; i = 1,2,\dots, \; k.$$

But, we have difficulty to prove that γ_i is limited for $i = 1,2,\dots, \; k$. Similarly, from Theorem 1.10, for every positive infinitesimal ε we have

$$\xi_i = \frac{p_i}{q} + \varepsilon £_i, \text{ with } |£_i| \le \frac{1}{q} = £ \; ; \; i = 1,2,\dots, \; k.$$

Also, if $\xi_1, \xi_2, \dots, \xi_k$ are linearly independent, we get the same result by using Theorem 1.12 whenever $\alpha_1 = \alpha_2 = \dots = \alpha_k = 0$. But we can not have the condition $\varepsilon q \simeq 0$.

The following definition gives a new sense to the simultaneous rational approximation of k numbers.

Definition 1.13. Let $(\xi_1, \xi_2, \dots, \xi_k)$ be a system of reals, with $k \ge 1$. The reals $(\xi_i)_{i=1,2,\dots,k}$ are said to be simultaneously approximable in the infinitesimal sense, if for every positive infinitesimal ε, there exist rational numbers $\left(\frac{p_i}{q}\right)_{i=1,2,\dots,k}$ such that

$$\begin{cases} \xi_i = \dfrac{p_i}{q} + \varepsilon £_i \\ \quad \varepsilon q \simeq 0, \end{cases} \; ; \; 1 \le i \le k, \tag{2}$$

where $(£_i)_{i=1,2,\dots,k}$ are limited numbers.

The real ε allows to control the error $\varepsilon £_i$ and the denominator q. In fact, from classical results of the simultaneous approximation, $\xi_i = \frac{p_i}{q} + e_i$ for $i = 1,2,\dots, k$. Then, for every positive infinitesimal ε, $\xi_i = \frac{p_i}{q} + \varepsilon £_i$, with $£_i = \frac{e_i}{\varepsilon}$. In Definition 1.13, we have added two conditions: $£_i$ is a limited for $i = 1,2,\dots,k$ and $\varepsilon q \simeq 0$.

Notation 1.14. *Let $S_A(\simeq 0)$ denote the set of all systems $(\xi_1, \xi_2, \dots, \xi_k)$, with $k \geq 1$ for which $(\xi_i)_{i=1,2,\dots,k}$ satisfy* (2).

In this paper, we will prove that $S_A(\simeq 0)$ is a non-empty set. Also, for a given system of reals $(\xi_0, \xi_1, \dots, \xi_\omega)$, with $\omega \simeq +\infty$ we ask if there is a necessary and a sufficient condition on the reals $(\xi_i)_{i=0,1,\dots,\omega}$ for which $(\xi_0, \xi_1, \dots, \xi_\omega) \in S_A(\simeq 0)$. We are in a position to give our main results.

2. MAIN RESULTS

To prove that $S_A(\simeq 0)$ is a non-empty set, we need the following lemma.

Lemma 2.1. *Let N, ω be two unlimited positive integers. Let ε be an infinitesimal positive real number. If $\varepsilon N^{\omega-1}$ is not infinitesimal, then there exists an integer $i_0 \in \{1, 2, \dots, \omega - 1\}$ such that $\varepsilon N^{\omega - i_0} \not\simeq 0$ and $\varepsilon N^{\omega-(i_0+1)} \simeq 0$.*

Proof. Let α be an appreciable number strictly less than $\varepsilon N^{\omega-1}$ which we may, because $\varepsilon N^{\omega-1} \not\simeq 0$. Since

$$0 \simeq \varepsilon N^{\omega-\omega} < \varepsilon N < \varepsilon N^2 < \cdots < \varepsilon N^{\omega-2} < \varepsilon N^{\omega-1} \not\simeq 0,$$

there exists an integer $s \in \{1, 2, \dots, \omega - 1\}$ such that

$$\varepsilon N^{\omega-(s+1)} \leq \alpha < \varepsilon N^{\omega-s}.$$

There are two cases to consider.

- $\varepsilon N^{\omega-(s+1)} \simeq 0$, Lemma 2.1 is proved by taking $i_0 = s$.
- $\varepsilon N^{\omega-(s+1)} \not\simeq 0$. Since $N \simeq +\infty$, we have

$$\varepsilon N^{\omega-(s+2)} = \frac{\varepsilon N^{\omega-(s+1)}}{N} \leq \frac{\alpha}{N} \simeq 0.$$

Also, Lemma 2.1 is proved by taking $i_0 = s + 1$. □

Theorem 2.2. $S_A(\simeq 0)$ *is a non-empty set.*

Proof. In the following proposition, we give a system containing an unlimited number of reals that satisfies (2). That is, we prove that $S_A(\simeq 0)$ contains many systems of the form $(\xi_0, \xi_1, \ldots, \xi_k)$, with k is an unlimited. □

Proposition 2.3. *Let* N, ω *be two unlimited positive integers. Then,*

$$\left(\frac{1}{N^{\omega}}, \frac{1}{N^{\omega-1}}, \ldots, \frac{1}{N}, 1\right) \in S_A(\simeq 0). \tag{3}$$

Proof. Let ε be an infinitesimal positive real number, there are two cases.

A) $\varepsilon N^{\omega} \simeq 0$. For every $i = 0, 1, \ldots, \omega$, we have

$$\begin{cases} \dfrac{1}{N^{\omega-i}} = \dfrac{N^i}{N^{\omega}} + \varepsilon.0 = \dfrac{p_i}{q} + \varepsilon £_i \\ \varepsilon N^{\omega} = \varepsilon q \simeq 0. \end{cases}$$

In this case, Proposition 2.3 is proved.

B) $\varepsilon N^{\omega} \not\simeq 0$. Here we distinguish two cases.

B.1) $\varepsilon N^{\omega} = a \simeq$ with a is an appreciable. In this case, we can write the system of (3) as follows:

$$\begin{pmatrix} \dfrac{1}{N^{\omega}} \\ \dfrac{1}{N^{\omega-1}} \\ \dfrac{1}{N^{\omega-2}} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{0}{N^{\omega-1}} + \varepsilon \dfrac{1}{a} \\ \dfrac{1}{N^{\omega-1}} + \varepsilon.0 \\ \dfrac{N}{N^{\omega-1}} + \varepsilon.0 \\ \vdots \\ \dfrac{N^{\omega-1}}{N^{\omega-1}} + \varepsilon.0 \end{pmatrix} = \begin{pmatrix} \dfrac{p_0}{q} + \varepsilon £_0 \\ \dfrac{p_1}{q} + \varepsilon £_1 \\ \dfrac{p_2}{q} + \varepsilon £_2 \\ \vdots \\ \dfrac{p_{\omega}}{q} + \varepsilon £_{\omega} \end{pmatrix},$$

where $\varepsilon q = \varepsilon N^{\omega-1} = \frac{\varepsilon N^{\omega}}{N} = \frac{a}{N} \simeq 0$. So, Proposition 2.3 is proved for this case.

B.2) $\varepsilon N^{\omega} \simeq +\infty$. In this case we also distinguish two cases.
B.2.1) The real $\varepsilon N^{\omega-1}$ is infinitesimal. Since $\frac{1}{\varepsilon N^{\omega}} \simeq 0$, it follows that

$$\begin{pmatrix} \frac{1}{N^{\omega}} \\ \frac{1}{N^{\omega-1}} \\ \frac{1}{N^{\omega-2}} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{0}{N^{\omega-1}} + \varepsilon \frac{1}{\varepsilon N^{\omega}} \\ \frac{1}{N^{\omega-1}} + \varepsilon .0 \\ \frac{N}{N^{\omega-1}} + \varepsilon .0 \\ \vdots \\ \frac{N^{\omega-1}}{N^{\omega-1}} + \varepsilon .0 \end{pmatrix} = \begin{pmatrix} \frac{p_0}{q} + \varepsilon £_0 \\ \frac{p_1}{q} + \varepsilon £_1 \\ \frac{p_2}{q} + \varepsilon £_2 \\ \vdots \\ \frac{p_{\omega}}{q} + \varepsilon £_{\omega} \end{pmatrix},$$

where $\varepsilon q = \varepsilon N^{\omega-1} \simeq 0$. Proposition 2.3 is proved.

B.2.2) The real $\varepsilon N^{\omega-1}$ is not infinitesimal. Let $i_0 \in \{1, 2, \dots, \omega - 1\}$ be the integer constructed in Lemma 2.1, then

$$\begin{pmatrix} \frac{1}{N^{\omega}} \\ \frac{1}{N^{\omega-1}} \\ \vdots \\ \frac{1}{N^{\omega-i_0}} \\ \frac{1}{N^{\omega-(i_0+1)}} \\ \frac{1}{N^{\omega-(i_0+2)}} \\ \vdots \\ \frac{1}{N} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{0}{N^{\omega-(i_0+1)}} + \varepsilon \frac{1}{\varepsilon N^{\omega}} \\ \frac{0}{N^{\omega-(i_0+1)}} + \varepsilon \frac{1}{\varepsilon N^{\omega-1}} \\ \vdots \\ \frac{0}{N^{\omega-(i_0+1)}} + \varepsilon \frac{1}{\varepsilon N^{\omega-i_0}} \\ \frac{1}{N^{\omega-(i_0+1)}} + \varepsilon .0 \\ \frac{N}{N^{\omega-(i_0+1)}} + \varepsilon .0 \\ \vdots \\ \frac{N^{\omega-(i_0+2)}}{N^{\omega-(i_0+1)}} + \varepsilon .0 \\ \frac{N^{\omega-(i_0+1)}}{N^{\omega-(i_0+1)}} + \varepsilon .0 \end{pmatrix} = \begin{pmatrix} \frac{p_0}{q} + \varepsilon £_0 \\ \frac{p_1}{q} + \varepsilon £_1 \\ \vdots \\ \frac{p_{i_0}}{q} + \varepsilon £_{i_0} \\ \frac{p_{i_0+1}}{q} + \varepsilon £_{i_0+1} \\ \frac{p_{i_0+2}}{q} + \varepsilon £_{i_0+2} \\ \vdots \\ \frac{p_{\omega-1}}{q} + \varepsilon £_{\omega-1} \\ \frac{p_{\omega}}{q} + \varepsilon £_{\omega} \end{pmatrix}$$

with $\varepsilon q = \varepsilon N^{\omega-(i_0+1)} \simeq 0$.

This completes the proof of Proposition 2.3. □

Lemma 2.4. *Let ω be an unlimited positive integer, and let $(\xi_0, \xi_1, \dots, \xi_\omega)$ be a system of reals satisfying the following properties:*

(a) $\xi_0 \simeq \xi_1 \simeq \cdots \simeq \xi_\omega$
(b) $\xi_{i+1} - \xi_i = d_i > 0$ *for* $i = 0, 1, \dots, \omega - 1$
(c) $\frac{d_i}{d_{i-1}} = a_i \simeq 1$ *for* $i = 1, 2, \dots, \omega - 1.$

Then, $(\xi_0, \xi_1, \dots, \xi_\omega) \notin S_A(\simeq 0)$.

Proof. Assume, by way of contradiction, that the reals $(\xi_i)_{i=0,1,\dots,\omega}$ are simultaneously approximable in the infinitesimal sense. In particular, for $\varepsilon = d_0 \simeq 0$ we have

$$\begin{cases} \xi_i = \dfrac{p_i}{q} + \varepsilon £_i \\ \quad \varepsilon q \simeq 0, \end{cases} \tag{4}$$

where $\frac{p_i}{q}$ is a rational and $£_i$ is a limited for every $i = 0,1, \dots, \omega$.

Let i_0 be an unlimited positive integer strictly less than ω and satisfying

$$i_0 < \frac{1}{N\varepsilon q} \tag{5}$$

for a given limited integer $N > 2$ (which we may, because $\frac{1}{\varepsilon q} = \frac{1}{d_0 q} \simeq +\infty$). Since the reals $(a_i)_{i=1,2,\dots,\omega-1}$ are all appreciable then, for any standard integer $n \geq 1$, the number $S_n = \sum_{i=1}^{n} a_1 a_2 \dots a_i$ is also an appreciable.

Next, consider the set

$$\left\{ n \in \{1, 2, \dots, \omega - 1\} ;\ 1 + \sum_{i=1}^{n} a_1 a_2 \dots a_i < i_0 \simeq +\infty \right\}, \tag{6}$$

which is internal and contains $\mathbb{N}^\sigma$. According to the Cauchy's Principle there exists an unlimited integer n_0 that satisfies (6).

On the other hand, since

$$\xi_{n_0} - \xi_0 = d_0 + d_1 + \ldots + d_{n_0-1} = \varepsilon \sum_{i=0}^{n_0-1} \frac{d_i}{d_0}, \tag{7}$$

and $\frac{d_i}{d_{i-1}} = a_i$, for $i = 1, 2, \ldots, \omega - 1$. From (4) and (7), we have

$$\begin{aligned} \xi_{n_0} - \xi_0 &= \varepsilon \left(1 + \sum_{i=1}^{n_0-1} a_1 a_2 \ldots a_i \right) \\ &= \frac{p_{n_0} - p_0}{q} + \varepsilon £. \end{aligned} \tag{8}$$

We use the fact that n_0 satisfies (6). Then from (5), (6), and (8) we get

$$(p_{n_0} - p_0) + \varepsilon q £ < \frac{1}{N}.$$

Since $\varepsilon q £ \simeq 0$, it follows that $p_{n_0} - p_0 < \frac{2}{N}$.

Now we prove that $p_{n_0} > p_0$. First, it suffices to prove that the number $1 + \sum_{i=1}^{n_0-1} a_1 a_2 \ldots a_i$ is unlimited. In fact, consider the following set

$$\left\{ m \in \mathbb{N} \, ; \; m \leq n_0 - 1 \; and \;\; 1 + \sum_{i=1}^{m} a_1 a_2 \ldots a_i > m \right\},$$

which is internal and contains $\mathbb{N}^{\sigma}$, because for all limited integers s we have

$$1 + \sum_{i=1}^{s} a_1 a_2 \ldots a_i = 1 + s + \phi_s > s,$$

where $\phi_s \simeq 0$ (positive or negative). From Cauchy's principle there exists an unlimited integer m_0 (with $m_0 \leq n_0 - 1$) such that

$$1 + \sum_{i=1}^{n_0-1} a_1 a_2 \ldots a_i \geq 1 + \sum_{i=1}^{m_0} a_1 a_2 \ldots a_i > m_0 \simeq +\infty.$$

We assume that $p_{n_0} = p_0$, by (8) we get

$$+\infty \simeq 1 + \sum_{i=1}^{n_0-1} a_1 a_2 \ldots a_i = £,$$

which is a contradiction. Therefore $p_{n_0} \neq p_0$. Moreover, if $p_{n_0} < p_0$, by using (8) again, we obtain

$$£ > \frac{p_0 - p_{n_0}}{\varepsilon q} \simeq +\infty,$$

because $\xi_{n_0} > \xi_0$. Which is a contradiction, since £ is limited. Recall that p_{n_0} and p_0 are positive integers, and since $p_{n_0} > p_0$ it follows that $\frac{2}{N} > 1$. Which leads to a contradiction with the hypothesis of $N > 2$. This completes the proof. □

Theorem 2.5. *Let ω be an unlimited positive integer, and let* $(\xi_0, \xi_1, \ldots, \xi_\omega)$ *be a system of reals. If* $°(\xi_0, \xi_1, \ldots, \xi_\omega)$ *contains a standard interval* $[a, b]$ *with* $a < b$ *then the reals* $(\xi_i)_{i=0,1,\ldots,\omega}$ *are not simultaneously approximable in the infinitesimal sense.*

That is, we will prove the necessary condition given by:

$$(\xi_0, \xi_1, \ldots, \xi_\omega) \in S_A(\simeq 0) \Rightarrow \forall a, b \in \mathbb{N}^\sigma : [a, b] \nsubseteq °(\xi_0, \xi_1, \ldots, \xi_\omega) \quad (\mathcal{N})$$

Proof. Since $°(\xi_0, \xi_1, \ldots, \xi_\omega)$ contains a standard interval $[a, b]$ with $a < b$, there exists a subsystem $(\xi_{i_0}, \xi_{i_1}, \ldots, \xi_{i_k}) \subset (\xi_0, \xi_1, \ldots, \xi_\omega)$ such that

$$\begin{cases} °(\xi_{i_0}, \xi_{i_1}, \ldots, \xi_{i_k}) = [a, b] \\ \xi_{i_0} < \xi_{i_1} < \ldots < \xi_{i_k}, \end{cases}$$

where $k \simeq +\infty$. We prove that $a \simeq \xi_{i_0} \simeq \xi_{i_1} \simeq \ldots \simeq \xi_{i_k} \simeq b$. In fact, suppose the contrary, *i.e.*, there exists $m \in \{1, 2, \ldots, k\}$ such that

$$\xi_{i_{m-1}} \not\simeq \xi_{i_m}.$$

Since $°\xi_{i_{m-1}} \neq °\xi_{i_m}$, it follows that

$$\frac{\xi_{i_{m-1}} + \xi_{i_m}}{2} \simeq \frac{{}^{\circ}\xi_{i_{m-1}} + {}^{\circ}\xi_{i_m}}{2} \in [a,b].$$

Which is a contradiction because the number $\frac{\xi_{i_{m-1}}+\xi_{i_m}}{2}$ does not belong to the system $\left(\xi_{i_0}, \xi_{i_1}, \dots, \xi_{i_k}\right)$.

Put $d_s = \xi_{i_{s+1}} - \xi_{i_s}$ for $s = 0, 1, \dots, k-1$, then $\max_{0\le s\le k-1}(d_s) \simeq 0$. Let γ be an unlimited real number such that

$$\lambda = \gamma \max_{0\le s\le k-1}(d_s) \simeq 0,$$

and this by using Robinson's Lemma.

Now, we choose a system of N elements $(\theta_r)_{r=0,1,\dots,N}$ among the numbers $(\xi_{i_s})_{s=0,1,\dots,k}$ as the following way

$$\theta_0 = \xi_{i_0}, \text{ and}$$

$$\theta_r = \xi_{i_{m_r}} \text{is the nearest element strictly less than } \theta_0 + r\,\lambda;\ r = 1,2,\dots,\ N.$$

where $m_r \in \{0, 1, \dots, k\}$ and N is an unlimited integer, with $N\lambda \simeq 0$. Then, we prove the conditions (**a**), (**b**) and (**c**) of the Lemma 2.4 for the new system $(\theta_0, \theta_1, \dots, \theta_N)$. In fact, from the construction of $(\theta_r)_{r=0,1,\dots,N}$ we see that

$$\theta_0 \simeq \theta_1 \simeq \cdots \simeq \theta_N \simeq \xi_{i_0} \text{ and } \theta_{r+1} - \theta_r = D_r > 0\,; 0 \le r \le N-1.$$

Thus, (**a**) and (**b**) are satisfied. For the proof of (**c**), we put

$$\delta_r = \xi_{i_0} + r\,\lambda - \theta_r\,;\ r = 0,1,\dots,\text{N}.$$

Then, $\delta_r \le \xi_{i_{m_r+1}} - \xi_{i_{m_r}} = d_{i_{m_r}}$, because $\xi_{i_{m_r}}$ is the nearest element strictly less than $\theta_0 + r\,\lambda$. Moreover, we have

$$\frac{\delta_r}{\lambda} = \frac{\delta_r}{\gamma \max_{0\le s\le k-1}(d_s)} \le \frac{1}{\gamma}\left(\frac{\delta_r}{d_{i_{m_r}}}\right) \le \frac{1}{\gamma} \simeq 0.$$

Therefore, for every $r = 0,1,\dots,\text{N}$, there exists an infinitesimal real number ϕ_r such that $\delta_r = \lambda\,\phi_r$. Hence

$\theta_{r+1} - \theta_r = \lambda - \delta_{r+1} + \delta_r = \lambda - \lambda\,\phi_{r+1} + \lambda\,\phi_r$; for $r = 0, 1, \dots, N-1$.

It follows for every $r \in \{1, 2, \dots, N-1\}$ that

$$\frac{D_r}{D_{r-1}} = \frac{\theta_{r+1} - \theta_r}{\theta_r - \theta_{r-1}} = \frac{1 - \phi_{r+1} + \phi_r}{1 - \phi_r + \phi_{r-1}} \simeq 1.$$

Using Lemma 2.4 we can also conclude that $(\theta_0, \theta_1, \dots, \theta_N) \notin S_A(\simeq 0)$ and therefore $(\xi_0, \xi_1, \dots, \xi_\omega) \notin S_A(\simeq 0)$. This completes the proof of Theorem 2.5. □

Corollary 2.6. *The set $S_A(\simeq 0)$ does not contain countable systems.*

Proof. Let $(\xi_0, \xi_1, \dots, \xi_\omega, \dots)$ be a countable system of reals. There exists a subsystem $(\xi_{i_0}, \xi_{i_1}, \dots, \xi_{i_k})$ satisfying the conditions of Lemma 2.4, with $k \simeq \infty$. Hence $(\xi_{i_0}, \xi_{i_1}, \dots, \xi_{i_k}) \notin S_A(\simeq 0)$, and therefore $(\xi_0, \xi_1, \dots, \xi_\omega, \dots) \notin S_A(\simeq 0)$. □

In the following result, for a real number x, let $\{x\}$ and $[x]$ denote the fractional part and the integer part of x, respectively.

Corollary 2.7. *Let ω be an unlimited positive integer. If $(\xi_0, \xi_1, \dots, \xi_\omega) \in S_A(\simeq 0)$ then, for every limited integer c, $^\circ(\{c\xi_0\}, \{c\xi_1\}, \dots, \{c\xi_k\})$ does not contain any standard interval $[a, b]$, with $a < b$.*

Proof. Suppose that there exists a subset of positive integers:

$$(i_0, i_1, \dots, i_k) \subset (0, 1, \dots, \omega), \text{ with } k \simeq +\infty,$$

and there is a limited integer c_0 such that $^\circ(\{c_0\xi_{i_0}\}, \{c_0\xi_{i_1}\}, \dots, \{c_0\xi_{i_k}\}) = [a, b]$ where a and b are standard real numbers ($a < b$), and we prove that $(\xi_0, \xi_1, \dots, \xi_\omega) \notin S_A(\simeq 0)$. In fact, from Theorem 2.5, we get

$$(\{c_0\xi_{i_0}\}, \{c_0\xi_{i_1}\}, \dots, \{c_0\xi_{i_k}\}) \notin S_A(\simeq 0). \tag{9}$$

It suffices to show that $(c_0\xi_{i_0}, c_0\xi_{i_1}, \dots, c_0\xi_{i_k}) \notin S_A(\simeq 0)$. Suppose the contrary. Then for every positive infinitesimal ε there exist rational numbers $\left(\frac{P_{i_s}}{Q}\right)_{s=0,1,\dots,k}$ such that

$$\begin{cases} \left\{c_0\xi_{i_s}\right\} = \dfrac{P_{i_s} - [c_0\xi_{i_s}]Q}{Q} + \varepsilon£ \\ \varepsilon Q \simeq 0 \end{cases} ;\ 0 \leq s \leq k,$$

because $\left\{c_0\xi_{i_s}\right\} = c_0\xi_{i_s} - [c_0\xi_{i_s}]$, for $s = 0,1,\dots,k$. Thus,

$$\left(\left\{c_0\xi_{i_0}\right\},\left\{c_0\xi_{i_1}\right\},\dots,\left\{c_0\xi_{i_k}\right\}\right) \in S_A(\simeq 0).$$

Which contradicts the expression (9).

Finally, since c_0 is a limited integer, we have $\left(\xi_{i_0},\xi_{i_1},\dots,\xi_{i_k}\right) \notin S_A(\simeq 0)$, and therefore $(\xi_0,\xi_1,\dots,\xi_\omega) \notin S_A(\simeq 0)$. This completes the proof. □

3. REMARKS AND EXAMPLES

In this section, we give certain remarks and examples about the necessary condition stated in Theorem 2.5.

Remark 3.1. *The converse of* $(\mathcal{N})$ *is false*.

In the following corollary, we give a system of real numbers $(\xi_0, \xi_1, \dots, \xi_\omega)$ with $\omega \simeq +\infty$, whose elements are not simultaneously approximable in the infinitesimal sense but its shadow is different from a standard interval $[a,b]$.

Corollary 3.2 (Counterexample). *Let f be the exponential function. For every unlimited positive integer ω, we have*

$$\left(\frac{1}{f(0)},\frac{1}{f(1)},\dots,\frac{1}{f(\omega)}\right) \notin S_A(\simeq 0). \tag{10}$$

Proof. Suppose that the reals of (10) are simultaneously approximable in the infinitesimal sense. Then, for $\varepsilon = \frac{1}{f(\omega)} \simeq 0$, there exist rational numbers $\left(\frac{p_i}{q}\right)_{i=0,1,\dots,\omega}$ such that

$$\begin{cases} \dfrac{1}{f(i)} = \dfrac{p_i}{q} + \varepsilon £_i \\ \varepsilon q \simeq 0, \end{cases}$$

where $£_i$ is a limited number for every $i = 0,\ 1, \ldots, \omega$.

Using Cauchy's principle, there exists an unlimited positive integer i_0 such that

$$f(i_0) < \frac{\gamma}{2}, \tag{11}$$

where $\gamma = \frac{1}{\varepsilon q} \simeq +\infty$. Since f is increasing, we have $i_0 < \omega$. In fact, if $i_0 \geq \omega$ it follows that $f(\omega) < \frac{f(\omega)}{2q}$. Which is impossible.

Now, we put $s_0 = \omega - i_0$. From the hypothesis, there exist $\frac{p_{s_0}}{q}, \frac{p_\omega}{q}$ such that

$$\frac{1}{f(s_0)} - \frac{1}{f(\omega)} = \varepsilon(f(i_0) - 1) = \frac{p_{s_0} - p_\omega}{q} + \ \varepsilon £. \tag{12}$$

Using (11) and (12), we get

$$(p_{s_0} - p_\omega\ ) + \ \varepsilon q £ < \frac{1}{2}. \tag{13}$$

It follows from (12) that $p_{s_0} \neq p_\omega$ because $f(i_0) \simeq +\infty$. Moreover, if $p_{s_0} < p_\omega$ then

$$£ > \frac{p_\omega - p_{s_0}}{\varepsilon q} \simeq +\infty.$$

which we may, because $\frac{1}{f(s_0)} > \frac{1}{f(\omega)}$. A contradiction, since £ is a limited. Thus, $p_{s_0} > p_\omega$. Finally, from (13) we have $1 \leq p_{s_0} - p_\omega < \frac{2}{3}$, since $\varepsilon q £ \simeq 0$. Which is impossible. □

Remark 3.3. Let f be the function of Corollary 3.2, we put $A = \left(\frac{1}{f(0)}, \frac{1}{f(1)}, \ldots, \frac{1}{f(\omega)}\right)$. Since f is standard, then $°A = A$ is not an interval.

Corollary 3.4 (An example of Theorem 2.5). *Let ω be an unlimited positive integer. Then,*

$$\left(\frac{1}{\omega},\frac{2}{\omega},\dots,\frac{\omega-1}{\omega},\frac{\omega}{\omega}\right) \notin S_A(\simeq 0).$$

Proof. It is clear that

$$^{\circ}\left(\frac{1}{\omega},\frac{2}{\omega},\dots,\frac{\omega-1}{\omega},\frac{\omega}{\omega}\right) = [0,1].$$

Thus we get the result by using Theorem 2.5. Moreover, for any standard interval $[a,b]$, with $a < b$, there exists a system of reals $(\xi_0,\xi_1,\dots,\xi_\omega)$, with $\omega \simeq +\infty$ such that $a \simeq \xi_0 \simeq \xi_1 \simeq \cdots \simeq \xi_\omega \simeq b$, and also from Theorem 2.5, $(\xi_0,\xi_1,\dots,\xi_\omega) \notin S_A(\simeq 0)$. □

Remark 3.5. Let $(\xi_0,\xi_1,\dots,\xi_k)$ be an arbitrary system of real numbers. From the proof of Corollary 2.7, it is clear that $(\xi_0,\xi_1,\dots,\xi_k) \in S_A(\simeq 0)$, if and only if $(\{\xi_0\},\{\xi_1\},\dots,\{\xi_k\}) \in S_A(\simeq 0)$, where $\{x\}$ represent the fractional part of x. We can therefore deduce that if

$$(\xi_0,\xi_1,\dots,\xi_k) \in S_A(\simeq 0)$$

then

$$(\xi_0,\xi_1,\dots,\xi_k,\{\xi_0\},\{\xi_1\},\dots,\{\xi_k\}) \in S_A(\simeq 0).$$

4. CONCLUSION

In this paper, we aim to give a necessary condition for a system of real numbers $(\xi_0,\xi_1,\dots,\xi_\omega)$ to be in $S_A(\simeq 0)$, where ω is an unlimited positive integer. However, a sufficient condition remains an open problem.

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